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POWER SERIES INVERSES OF THE UNIVERSAL FORM OF KEPLER'S EQUATION

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ABSTRACT

Two power series inverses of the universal form of Kepler's equation are derived with simple recursion formulas for the coefficients. The first series is in powers of time, the second in powers of $\alpha = 1/a$, where a is the semimajor axis. The second series is particulary useful for nearly parabolic orbits, since Kepler's equation is easily solved when $\alpha = 0$, i.e., when the motion is parabolic.

LIST OF SYMBOLS

a	semi-major axis of orbit
e	eccentricity of orbit
p	semilatus rectum of orbit
$\vec{r}$	orbital position vector at time t
r	magnitude of the orbital position vector at time t
$\vec{r}_o$	orbital position vector at time t_o
r_o	magnitude of the orbital position vector at time t_o
$\dot{r}_o$	time derivative of r at time t_o
t	time
t_o	reference time
θ	true anomaly of orbit
μ	gravitational constant times the mass of the attracting body
$\vec{v}$	orbital velocity vector at time t
$\vec{v}_o$	orbital velocity vector at time t_o
v_o	magnitude of $\vec{v}_o$

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INTRODUCTION

The universal form of Kepler's Equation is [1]

$$A\sigma + B\gamma + r_o\beta = \mu^{1/2}(t - t_o), \tag{1}$$

$$A = 1 - \alpha r_o \ ,$$

$$B = r_o \dot{r}_o / \mu^{1/2},$$

$$\alpha = \frac{1}{a} = \frac{2}{r_o} - \frac{v_o^2}{\mu} \ ,$$

$$\sigma = \beta^3 S(\alpha\beta^2), \tag{2}$$

$$\gamma = \beta^2 C(\alpha\beta^2) \ . \tag{3}$$

The S and C functions are defined by the series

$$S(x) = \sum_{i=0}^{\infty} \frac{(-x)^i}{(2i + 3)!} \ , \tag{4}$$

$$C(x) = \sum_{i=0}^{\infty} \frac{(-x)^i}{(2i+2)!} . \tag{5}$$

Given the time t and the position and velocity at time t_o, β is found from equation (1). The position and velocity can then be found at time t from

$$\vec{r} = (1 - \gamma/r_o)\,\vec{r}_o + (t - t_o - \sigma/\mu^{1/2})\,\vec{v}_o, \tag{6}$$

$$\vec{v} = \frac{\mu^{1/2}}{r_o r}(\alpha\sigma - \beta)\,\vec{r}_o + (1 - \gamma/r)\,\vec{v}_o, \tag{7}$$

$$r = A\gamma - B(\alpha\sigma - \beta) + r_o. \tag{8}$$

Chebyshev approximations of the functions S(x) and C(x) have been obtained [2] which reduce significantly the computation times required for their evaluation when compared to that required by the series (4) and (5).

We have obtained a power series solution of equation (1) for β as a function of $t - t_o$. The procedure simultaneously produces series for σ and γ, which are needed in the updating equations (6), (7), and (8). The recursion formulas for the coefficients in these series are extremely simple, making possible a computer program which is not only fast but requires very few computer storage locations.

If we let $\alpha = 0$ in equations (2) and (3) for σ and γ, equation (1) becomes a cubic in β:

$$\frac{1}{6}A\beta^3 + \frac{1}{2}B\beta^2 + r_o\beta - \mu^{1/2}(t - t_o) = 0. \tag{9}$$

Our second series is a perturbative power series solution of (1) for β as a function of α with t fixed, i.e.,

$$\beta = \sum_{i=0}^{\infty} \beta_i \alpha^i ,$$

where β_o is the solution of (9).

A sufficient condition for (9) to have only one real root is that the quadratic obtained by differentiating (9) have no real roots.

The discriminant of this quadratic is

$$D = B^2 - 2 r_o A$$

$$= 2 \frac{r_o^2}{a} - 2 r_o + \frac{r_o^2 \dot{r}_o^2}{\mu}$$

$$= \frac{r_o^2}{a} - p,$$

where p is the semilatus rectum of the orbit. Substituting

$$r_o = \frac{p}{1 + e \cos \theta}$$

into the expression for D gives

$$D = - \frac{ep \, (e + 2 \cos \theta + e^2 \cos^2 \theta)}{(1 + e \cos \theta)^2} .$$

Thus the quadratic will have no real roots provided

$$e + e^2 \cos^2 \theta + 2 \cos \theta \geq 0, \tag{10}$$

and this is a sufficient condition for (9) to have only one real root.

Obviously (10) holds for all hyperbolic and parabolic orbits and for all elliptic orbits such that $|\theta| \leq 90°$. For $\theta > 90°$ (9) has three real roots for some elliptic orbits. We will not pursue the matter further, however, since the universal form of Kepler's equation is not needed for $\theta > 90°$.

In reference [3] the initial position vector $\vec{r}_o$ is chosen at periapsis, and after finding β_o from the cubic equation, a correction is found by the Newton-Raphson procedure.

In reference [4] recursion formulas are derived for the coefficients of three power series inverses of the elliptic form of Kepler's equation: (1) a series in powers of time, (2) a series in powers of eccentricity, (3) a bivariate series in powers of two small parameters.

POWER SERIES IN TIME

We define the time from epoch as $\tau = t - t_o$, so that from (1) we have

$$A\sigma + B\gamma + r_o\beta = \sqrt{\mu}\,\tau. \tag{11}$$

Then from the identities

$$\frac{dS}{dx} = \frac{1}{2x}\left[C(x) - 3S(x)\right],$$

$$\frac{dC}{dx} = \frac{1}{2x}\left[1 - xS(x) - 2C(x)\right],$$

which are easily established from the definitions (4) and (5), we obtain the equations

$$\dot{\sigma} = \gamma\dot{\beta}, \tag{12}$$

$$\dot{\gamma} = (\beta - \alpha\sigma)\,\dot{\beta}, \tag{13}$$

where the dot indicates differentiation with respect to time.

Let $$\beta = \sum_{i=0}^{\infty} \beta_i\,\tau^i,$$

$$\sigma = \sum_{i=0}^{\infty} \sigma_i\,\tau^i,$$

$$\gamma = \sum_{i=0}^{\infty} \gamma_i \tau^i .$$

Then
$$\dot{\beta} = \sum_{i=0}^{\infty} (i+1) \beta_{i+1} \tau^i ,$$

$$\dot{\sigma} = \sum_{i=0}^{\infty} (i+1) \sigma_{i+1} \tau^i ,$$

$$\dot{\gamma} = \sum_{i=0}^{\infty} (i+1) \gamma_{i+1} \tau^i .$$

Substituting the last six equations into (11), (12) and (13) and equating coefficients of like powers of τ, we obtain

$$A\sigma_{i+1} + B\gamma_{i+1} + r_o \beta_{i+1} = 0 ,$$

$$(i+1) \sigma_{i+1} = \sum_{j=0}^{i} (j+1) \beta_{j+1} \gamma_{i-j} ,$$

$$(i+1) \gamma_{i+1} = \sum_{j=0}^{i} (j+1) \beta_{j+1} \beta_{i-j} - \alpha \sum_{j=0}^{i} (j+1) \beta_{j+1} \sigma_{i-j} ,$$

all valid for $i \geq 1$.

However, noting that

$$\beta_o = \sigma_o = \gamma_o = 0 \ , \tag{14}$$

$$\beta_1 = \sqrt{\mu}, \ \gamma_1 = \sigma_1 = 0 \ , \tag{15}$$

$$\beta_2 = -\frac{B\mu}{2r_o} \ \gamma_2 = \frac{1}{2}\mu \ , \ \sigma_2 = 0 \ , \tag{16}$$

we have the following recursion formulas for $i \geq 2$,

$$\delta_j = j \ \beta_j \ ,$$

$$(i+1)\,\sigma_{i+1} = \sum_{j=1}^{i-1} \delta_j \ \gamma_{i-j+1} \ ,$$

$$(i+1)\,\gamma_{i+1} = \sum_{j=1}^{i} \delta_j \ \beta_{i-j+1} - \alpha \sum_{j=1}^{i-2} \delta_j \ \sigma_{i-j+1} \ , \tag{17}$$

$$r_o \ \beta_{i+1} = -A\sigma_{i+1} - B\gamma_{i+1} \ .$$

If we wish merely to calculate updated values of β, σ, or γ, we let

$$\beta_1 = \tau\sqrt{\mu} \ ,$$

$$\beta_2 = -\frac{B\mu\tau^2}{2r_o} ,$$

$$\gamma_2 = \frac{1}{2} \mu\tau^2$$

in equations (15) and (16). The recursion formulas (28) then produce coefficients such that the series become

$$\beta = \sum_{i=0}^{\infty} \beta_i ,$$

$$\sigma = \sum_{i=0}^{\infty} \sigma_i ,$$

$$\gamma = \sum_{i=0}^{\infty} \gamma_i .$$

POWER SERIES IN α

Equations (2) and (3) are equivalent to [1]

$$\alpha^{3/2}\sigma = \alpha^{1/2}\beta - \sin(\alpha^{1/2}\beta) ,$$

for $\alpha > 0$ and

$$(-\alpha)^{3/2}\sigma = \sinh\left((-\alpha)^{1/2}\beta \right) - (-\alpha)^{1/2}\beta ,$$

for $\alpha < 0$ and

$$\alpha\gamma = 1 - \cosh\left((-\alpha)^{1/2}\beta\right)$$

for all α.

Differentiating these equations we obtain,

$$\frac{3}{2}\sigma + \alpha\sigma' = p\gamma \quad , \tag{18}$$

$$\gamma + \alpha\gamma' = p\,\delta \tag{19}$$

for all α, where the prime denotes differentiation with respect to α and

$$\delta = \beta - \alpha\sigma \quad , \tag{20}$$

$$p = \frac{1}{2}\beta + \alpha\beta' \quad . \tag{21}$$

To this set we add Kepler's equation

$$A\sigma + B\gamma + r_o\beta = C \quad , \tag{22}$$

$$C = \mu^{1/2}(t - t_o) \quad .$$

Let $$\beta = \sum_{i=0}^{\infty} \beta_i\,\alpha^i \quad ,$$

$$\sigma = \sum_{i=0}^{\infty} \sigma_i \, \alpha^i ,$$

$$\gamma = \sum_{i=0}^{\infty} \gamma_i \, \alpha^i ,$$

$$\delta = \sum_{i=0}^{\infty} \delta_i \, \alpha^i ,$$

$$p = \sum_{i=0}^{\infty} p_i \, \alpha^i .$$

Substituting these series into (18), (19), and (22) and equating coefficients of like powers of α, we have, for $i > 1$,

$$\frac{1}{2}\beta_o^2 \left(i + \frac{1}{2}\right) \beta_i + \frac{1}{2}\beta_o \gamma_i - \left(i + \frac{3}{2}\right) \sigma_i = - \sum_{j=1}^{i-1} p_j \gamma_{i-j} , \tag{23}$$

$$\beta_o \beta_i - \gamma_i = \left(\frac{1}{2}\beta_o \sigma_{i-1} - \sum_{j=1}^{i-1} p_j \, \delta_{i-j} \right) \Big/ (i + 1) , \tag{24}$$

$$r_o \, \beta_i + B\gamma_i + A\sigma_i = 0 . \tag{25}$$

Multiplying (24) by $\frac{1}{2}\beta_o$ and adding to (23) gives σ_i in terms of β_i. Substituting that result with (24) into (25) yields a recursion formula for the β_i. We also substitute the series into (18) and (19). Our final result is

$$p_o = \frac{1}{2}\beta_o , \quad \delta_o = \beta_o , \quad \gamma_o = p_o \delta_o , \quad \sigma_o = \frac{2}{3} p_o \gamma_o ,$$

where β_o is the real root of

$$\frac{1}{6}A\beta^3 + \frac{1}{2}B\beta^2 + r_o\beta - C = 0 \quad ,$$

$$S = r_o + B\beta_o + A\gamma_o , \; b = \frac{1}{2} p_o\sigma_o \quad ,$$

$$\beta_1 = b(B + \frac{2}{5} p_o A)/S \quad ,$$

$$p_1 = \frac{3}{2}\beta_1 , \; \delta_1 = \beta_1 - \sigma_o , \; \gamma_1 = \beta_o\beta_1 - b , \; \sigma_1 = \gamma_o\beta_1 - \frac{2}{5} p_o b \quad ,$$

and for $i > 1$,

$$F = \left(p_o\sigma_{i-1} - \sum_{j=1}^{i-1} p_j \, \delta_{i-j} \right) \Big/ (i + 1) \quad ,$$

$$G = \left(p_o F - \sum_{j=1}^{i-1} p_j \gamma_{i-j} \right) \Big/ (i + \frac{3}{2}) \quad ,$$

$$\beta_i = (BF + AG)/S \quad ,$$

$$\gamma_i = \beta_o\beta_i - F \quad ,$$

$$\sigma_i = \gamma_o \beta_i - G \quad ,$$

$$p_i = (i + \frac{1}{2}) \beta_i \quad ,$$

$$\delta_i = \beta_i - \sigma_{i-1} \quad .$$

We note that the divisor S in the equation for β_i can be written

$$S = \frac{1}{2} A\beta_o^2 + B\beta_o + r_o \quad ,$$

and this cannot be zero if we conform to the restrictions suggested in the Introduction.

Again, if we wish to calculate updated values of β, σ, or γ, we let

$$F = \left(\alpha p_o \sigma_{i-1} - \sum_{j=1}^{i-1} p_j \, \delta_{i-j} \right) \Big/ (i + 1)$$

$$\delta_i = \beta_i - \alpha\sigma_{i-1}$$

so that the recursion formulas produce coefficients such that the series become

$$\beta = \sum_{i=0}^{\infty} \beta_i \quad ,$$

$$\sigma = \sum_{i=0}^{\infty} \sigma_i \quad ,$$

$$\gamma = \sum_{i=0}^{\infty} \gamma_i \quad .$$

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